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The optimal periodic motions of a two-mass system in a resistant medium☆

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ABSTRACT

The rectilinear motions of a two-mass system, consisting of a container and an internal mass, in a medium with resistance, are considered. The displacement of the system as a whole occurs due to periodic motion of the internal mass with respect to the container. The optimal periodic motions of the system, corresponding to the greatest velocity of displacement of the system as a whole, averaged over a period, are constructed and investigated using a simple mechanical model. Different laws of resistance of the medium, including linear and quadratic resistance, isotropic and anisotropic, and also a resistance in the form of dry-friction forces obeying Coulomb's law, are considered.

A number of publications have been devoted to the principle of motion considered in connection with the dynamics of mobile robots and underwater equipment (see, for example, Refs. 1–3); the principle has been realized in a number of experimental models of mobile robots.

1. Formulation of the problem

Consider a two-mass system, consisting of two rigid bodies: a main body (a container) of mass M and an internal body of mass m (Fig. 1). For brevity we will call these bodies "body M" and "mass m" respectively. Body M is in a resistant medium, while the mass m moves inside the body M. Both bodies move translationally and rectilinearly. We will investigate the periodic motions of the mass m relative to the body M, in which the system moves as a whole in a resistant medium.

We will denote by x and v the absolute coordinate and velocity of the body M, and by ξ, u and w the displacement, velocity and acceleration of the mass m with respect to the body M.

The kinematic equations of motion of the mass m with respect to the body M have the form

$$\dot{\xi} = u, \quad \dot{u} = w \tag{1.1}$$

The dots denote derivatives with respect to time t.

We will denote by R the resistance force with which the external medium acts on the body M, and by F the force with which the internal mass m acts on this body. Then the equations of the dynamics of the system can be written in the form

$$\dot{x} = \upsilon, \quad M\dot{\upsilon} = R + F, \quad m(\dot{\upsilon} + w) = -F$$

Adding the last two equations and introducing the notation

$$\mu = m/(M+m), \quad -R/(M+m) = r(\upsilon) \tag{1.2}$$

we obtain the equations of motion of the body M

$$\dot{x} = \upsilon, \quad \dot{\upsilon} = -\mu w - r(\upsilon) \tag{1.3}$$

Note that, for any resistant media, $\upsilon r(\upsilon) \geq 0$.

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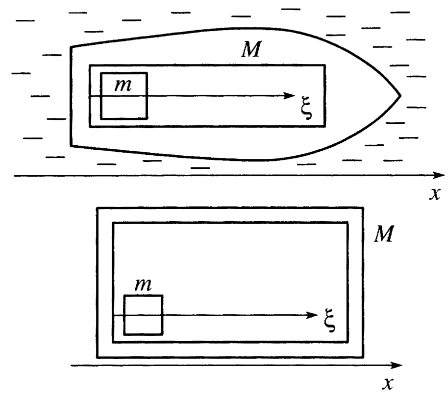

Fig. 1.

In particular, for a resistance law that is linear and anisotropic, i.e., dependent on the direction of motion, we have

$$r(v) = k_+ v \text{ when } v \geq 0, \quad r(v) = k_- v \text{ when } v \leq 0 \tag{1.4}$$

where k_+ and k_- are positive coefficients of resistance; in the isotropic case $k_+ = k_-$.

In the case of a quadratic anisotropic law of resistance we assume

$$r(v) = \kappa_+ |v| v \text{ when } v \geq 0, \quad r(v) = \kappa_- |v| v \text{ when } v \leq 0 \tag{1.5}$$

where κ_+ and κ_- are positive coefficients.

For dry friction, which obeys Coulomb's law, the function $r(v)$ is given in the form

$$r(v) = f_+ g \text{ when } v > 0, \quad r(v) = -f_- g \text{ when } v < 0 \tag{1.6}$$

where g is the acceleration due to gravity, and f_+ and f_- are constant coefficients of friction. If the following conditions are satisfied

$$-f_+ g \leq \mu w \leq f_- g \tag{1.7}$$

and the body M is at rest ($v = 0$), the state of rest will be maintained. In the isotropic case $f_+ = f_-$.

The motion of the mass m relative to the body M is assumed to be periodic with period T and confined inside a specified range of length L, i.e.

$$0 \leq \xi(t) \leq L \tag{1.8}$$

where $L > 0$ is a specified quantity. Condition (1.8) reflects practical limitations, imposed on the motion of the internal mass: the presence of a cavity or stops.

Without loss of generality, we can assume that at the beginning and end of the period the mass m is at the left end of the interval (1.8). The periodicity conditions then take the form

$$\xi(0) = \xi(T) = 0, \quad u(0) = u(T) = 0 \tag{1.9}$$

At a certain instant $\theta \in (0, T)$ the mass m reaches the right end of the interval (1.8), so that $\xi(\theta) = L$.

The motion of the system as a whole is controlled by the relative motion of the mass m, in which case the functions $\xi(t)$, $u(t)$ and $w(t)$ play the role of controls. These functions are related by Eqs. (1.1) and satisfy constraints (1.8) and periodicity conditions (1.9).

We will seek those relative periodic motions for which

1) the absolute velocity of the body M will be periodic, i.e. the following condition is satisfied

$$v(0) = v(T) = v_0 \tag{1.10}$$

2) the average velocity of the system as a whole

$$V = \Delta x / T; \quad \Delta x = x(T) - x(0) \tag{1.11}$$

will be a maximum.

Here v_0 is an as yet unknown constant, to be determined. The optimal periodic motions will be sought in a certain class of motions, defined below.

Note that similar problems were considered previously in Refs. 4–6. It was assumed in Ref. 4 that the mass m is not inside the body M, and, like the container itself, is subjected to the action of dry-friction forces. In the case of dry-friction forces, the additional condition $v_0 = 0$ in relation (1.10) was assumed in Refs. 5 and 6. In view of this constraint, the maximum average velocity (1.11), calculated previously in Refs. 5 and 6, turns out to be less than that obtained in the present paper.

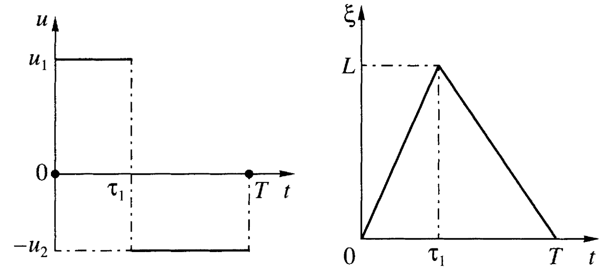

Fig. 2.

2. A linear resistance

Note that the resistance law (1.4) is non-linear in the anisotropic case ($k_+ \neq k_-$). We will consider the linear isotropic case, assuming $k_+ = k_- = k$. We substitute expressions (1.4) into Eq. (1.3) and integrate them over the period [0, T], also taking Eq. (1.1) into account. We obtain

$$v(T) - v(0) = -\mu[u(T) - u(0)] - k[x(T) - x(0)]$$

By virtue of the periodicity conditions (1.9) and (1.10) for the velocities $u(t)$ and $v(t)$ we obtain $x(T) = x(0)$. By condition (1.11) we therefore have $V = 0$.

Consequently, in the case of an isotropic linear resistance for any periodic relative motion of the internal mass m, the system as a whole cannot move; it will execute oscillations about a certain mean position.

3. Relative motion

We will confine our consideration to one simple class of relative motions of the internal mass m, which was called two-phase motion in Refs. 5 and 6. We will assume that the period [0,T] consists of two parts, in which the relative velocity $u(t)$ is constant. We will denote by τ_1 and τ_2 the durations of these parts, and by u_1 and ($-u_2$) the values of the constant velocity in these parts. Hence we have

$$u(t) = u_1 \text{ when } t \in (0, \tau_1), \quad u(t) = -u_2 \text{ when } t \in (\tau_1, T); \qquad T = \tau_1 + \tau_2 \tag{3.1}$$

Here u_1 and u_2 are positive constants. Note that the signs of the constant velocity in Eqs. (3.1) are determined by conditions (1.8) and (1.9). It follows from these that $u(t) > 0$ for small $t > 0$ and $u(t) < 0$ for t close to T. Since the velocity $u(t)$ has jumps at the beginning and end of the period [0, T], at these instants the function $u(t)$ may be defined differently, but this does not affect the motion of the system. For convenience and without loss of generality we will assume that conditions (1.9) are satisfied. Graphs of $u(t)$ and $\xi(t)$ are shown in Fig. 2.

According to relations (1.1) and (3.1), the relative acceleration of the mass m is defined as follows

$$w(t) = u_1\delta(t) - (u_1 + u_2)\delta(t - \tau_1) + u_2\delta(t - T) \tag{3.2}$$

where $\delta(t)$ is the delta function.

The two-phase relative motion (3.1) considered is governed by two constants: u_1 and u_2 or τ_1 and τ_2. Since during each of the two phases the mass m traverses the interval (1.8) (in opposite directions, see Fig. 2), we have the following relation between these constants

$$\tau_1 = Lu_1^{-1}, \quad \tau_2 = Lu_2^{-1}, \quad T = L(u_1^{-1} + u_2^{-1}), \quad \theta = \tau_1 \tag{3.3}$$

If the relative velocity $u(t)$ is bounded above by the constant U, we have the constraints

$$0 < u_i \leq U, \quad i = 1, 2 \tag{3.4}$$

imposed on the constants u_1 and u_2.

Hence, the problem of finding the optimum periodic solution reduces to determining the constants u_1 and u_2, which satisfy constraints (3.4) and which make the average velocity (1.11) a maximum.

4. A non-linear resistance

We substitute expression (3.2) into Eq. (1.3) and denote the values of the velocity $v(t)$ directly before its jumps at the instants τ_1 and T by v_1 and v_2, respectively. We have

$$v(0) = v_0, \quad v(+0) = v_0 - \mu u_1, \quad v(\tau_1 - 0) = v_1$$
$$v(\tau_1 + 0) = v_1 + \mu(u_1 + u_2), \quad v(T - 0) = v_2, \quad v(T) = v_2 - \mu u_2 = v_0 \tag{4.1}$$

The last equality follows from the periodicity condition (1.10).

Note that the velocity $v(t)$, in view of Eq. (1.3), cannot change sign inside the intervals $(0,\tau_1)$ and (τ_1, T). A change in the sign of $v(t)$ only occurs at the instant $t = \tau_1$, where, as follows from relations (4.1), we have the inequalities

$$v(t) \leq 0 \text{ when } t \in (0, \tau_1), \quad v(t) \geq 0 \text{ when } t \in (\tau_1, T) \tag{4.2}$$

Bearing relations (4.1) in mind, we obtain from Eq. (1.3)

$$\int\limits_{v_0-\mu u_1}^{v_1}\frac{dv}{r(v)} = -\tau_1, \qquad \int\limits_{v_1+\mu(u_1+u_2)}^{v_0+\mu u_2}\frac{dv}{r(v)} = -\tau_2 \tag{4.3}$$

For specified parameters of the relative motion u_1 and u_2 or τ_1 and τ_2, connected by relations (3.3), the two Eq. (4.3) serve to determine the two quantities v_0 and v_1. Determining these quantities, we thereby construct the periodic solution $v(t)$ for the specified relative motion of the mass m.

In order to obtain the optimal motion, we must determine the maximum of the average velocity (1.11) with respect to the parameters u_1 and u_2 (or τ_1 and τ_2) taking relations (3.3) and constraints (3.4) into account. We will carry out these calculations separately for the cases of piecewise-linear resistance (1.4) and quadratic resistance (1.5), and also for the case of dry friction (1.6).

5. Piecewise-linear resistance

We substitute expressions (1.4) into the integrals (4.3) and evaluate these integrals, taking into account the signs (4.2) of the function $v(t)$ in the corresponding intervals. Introducing the notation

$$e_1 = \exp(-k_-\tau_1), \quad e_2 = \exp(-k_+\tau_2) \tag{5.1}$$

we obtain the equations

$$v_1 = (v_0-\mu u_1)e_1, \quad v_0+\mu u_2 = [v_1+\mu(u_1+u_2)]e_2$$

Hence we find

$$v_0 = \mu[u_1e_2(1-e_1)-u_2(1-e_2)](1-e_1e_2)^{-1}$$
$$v_1 = -\mu(u_1+u_2)e_1(1-e_2)(1-e_1e_2)^{-1} \tag{5.2}$$

Integrating Eqs. (1.3) and (1.4) and taking the initial conditions (4.1) and relations (5.2) into account, we obtain

$$v(t) = -\mu(u_1+u_2)(1-e_2)(1-e_1e_2)^{-1}\exp(-k_-t) \quad \text{when} \quad t\in(0,\tau_1)$$

$$v(t) = \mu(u_1+u_2)e_2(1-e_1)(1-e_1e_2)^{-1}\exp[k_+(T-t)] \quad \text{when} \quad t\in(\tau_1, T) \tag{5.3}$$

Evaluating the integral of the velocity (5.3) over a period, we determine the displacement

$$\Delta x = \int\limits_0^T v(t)dt = \mu(u_1+u_2)\nu; \quad \nu = (1-e_1)(1-e_2)(1-e_1e_2)^{-1}(k_+^{-1}-k_-^{-1})$$

Expressing the parameters u_1 and u_2 in the expression obtained in terms of τ_1 and τ_2 using equalities (3.3), we obtain

$$\Delta x = \mu LT\nu\tau_1^{-1}\tau_2^{-1} \tag{5.4}$$

Note that since $e_1<1$ and $e_2<1$ according to the notation (5.1), the total displacement Δx of the body M in a period is positive when $k_+<k_-$ and negative when $k_+>k_-$. This result is quite natural. We will henceforth assume that $k_+<k_-$, so that $\Delta x>0$.

The average velocity of the body M over a period is given by Eqs. (1.11) and (5.4). We have

$$V = \mu L\nu\tau_1^{-1}\tau_2^{-1} \tag{5.5}$$

We will now optimize the average velocity V of the body M with respect to the parameters of relative motion of the mass m.

For given μ, L, k_+ and k_- the velocity V depends on two positive parameters τ_1 and τ_2. Constraints (3.4), imposed on u_1 and u_2, are transformed in accordance with relations (3.3) into the following inequalities

$$\tau_1\ge\tau_0, \quad \tau_2\ge\tau_0 \tag{5.6}$$

where we have introduced the notation

$$\tau_0 = L/U>0 \tag{5.7}$$

We will estimate the partial derivatives of the function $V(\tau_1,\tau_2)$ with respect to τ_1 and τ_2. To do this we note that the function $V(\tau_1,\tau_2)$, defined by formula (5.5), can be represented in the form

$$V(\tau_1,\tau_2) = V_1(y,\tau_2)V_2(\tau_2) \tag{5.8}$$

where we have put

$$V_1(y,\tau_2) = \frac{e^y-1}{y(e^y-e_2)}, \quad y = k_-\tau_1 \tag{5.9}$$

while $V_2(\tau_2)$ is a certain positive function, independent of τ_1.

Differentiating the function V_1 with respect to y, we obtain

$$\frac{\partial V_1}{\partial y} = \frac{e^y(y - e_2y - e^y + 1 + e_2 - e_2e^{-y})}{y^2(e^y - e_2)^2} < \frac{e^y(y - e_2ye^{-y} - e^y + 1 + e_2 - e_2e^{-y})}{y^2(e^y - e_2)^2}$$

$$= \frac{e^y(1 - e_2e^{-y})(y - e^y + 1)}{y^2(e^y - e_2)^2} = \frac{y + 1 - e^y}{y^2(e^y - e_2)} < 0$$

Hence, in view of relations (5.8) and (5.9), it follows that $\partial V/\partial\tau_1 < 0$. Since the function $V(\tau_1,\tau_2)$ is invariant under the replacement $\tau_1 \to \tau_2$, $\tau_2 \to \tau_1$, it hence also follows that $\partial V/\partial\tau_2 < 0$.

Consequently, the function $V(\tau_1,\tau_2)$ decreases as each of the arguments τ_1 and τ_2 increases, and it reaches its required maximum for the least values of the arguments, allowed by conditions (5.6), i.e. when

$$\tau_1 = \tau_2 = \tau_0 = L/U \tag{5.10}$$

Here, by virtue of relations (3.3), we have

$$u_1 = u_2 = U, \quad T = 2\tau_0 \tag{5.11}$$

Hence, the optimum relative motion of the mass m is motion with the maximum possible velocity U first from the point $\xi = 0$ to the point $\xi = L$, and then with the same value of the velocity in the reverse direction: from the point $\xi = L$ to the point $\xi = 0$.

The maximum possible average velocity of the body M is given by relations (5.5) and (5.10) in the form

$$V_{\max} = \mu U^2L^{-1}(1 - e_1)(1 - e_2)(1 - e_1e_2)^{-1}(k_+^{-1} - k_-^{-1}) \tag{5.12}$$

The quantities e_1 and e_2 are defined by expressions (5.1).

When $k_-L/U \ll 1$, formula (5.12) is simplified. Since $k_+ < k_-$, we also have $k_+L/U \ll 1$, and relation (5.12) takes the form

$$V_{\max} = \mu U(k_- - k_+)/(k_- + k_+)$$

6. Quadratic resistance

We will now consider the case of quadratic resistance (1.5). For simplicity we will confine ourselves to an isotropic resistance and put $\kappa_+ = \kappa_- = \kappa$ in relations (1.5). We evaluate the integrals (4.3) for the case in question, taking into account the signs of the velocity $v(t)$ in accordance with inequalities (4.2). We obtain

$$\frac{v_1 + \mu u_1 - v_0}{v_1(v_0 - \mu u_1)} = \kappa\tau_1, \quad \frac{v_1 + \mu u_1 - v_0}{(v_0 + \mu u_2)(v_1 + \mu u_1 + \mu u_2)} = \kappa\tau_2 \tag{6.1}$$

Relations (6.1) for specified u_1 and u_2 form a system of two algebraic equations in the parameters v_0 and v_1.

To simplify the calculations we will seek the solution of system (6.1) for which $v_0 = 0$; we thereby impose a certain additional constraint on the parameters u_1 and u_2. Using relations (3.3), we obtain from the first equation of (6.1) in the case considered

$$v_0 = 0, \quad v_1 = -\mu u_1(1 + \mu q)^{-1} \tag{6.2}$$

Here we have introduced the following notation for the dimensionless parameter

$$\kappa L = q \tag{6.3}$$

From the second Eq. (6.1), taking Eqs. (6.2) and (6.3) into account, we determine the relation between the parameters u_1 and u_2 in the form

$$u_2 = (1 - \mu q)(1 + \mu q)^{-1}u_1 \qquad (\mu q < 1) \tag{6.4}$$

Here and below we will assume that the condition $\mu q < 1$ is satisfied; otherwise solution (6.2) does not exist when $u_2 > 0$.

We integrate the second equation of (1.3) in the case of quadratic resistance (1.5) when $\kappa_+ = \kappa_- = \kappa$ and for initial conditions specified by relations (4.1) and (6.2). Also using equalities (6.3) and (6.4), we obtain the velocity of the body M in the form

$$v(t) = -\frac{\mu u_1}{1 + \mu u_1\kappa t} \quad \text{when } t \in (0, \tau_1)$$

$$v(t) = \frac{\mu u_1}{1 + \mu q + \mu u_1\kappa(t - \tau_1)} \quad \text{when } t \in (\tau_1, T) \tag{6.5}$$

We will determine the displacement Δx of the body M in the period $[0, T]$ corresponding to Eq. (1.3), for which we integrate the velocity $v(t)$, defined by formulae (6.5). After simplification we obtain

$$\Delta x = \int_0^T \upsilon(t)dt = -\kappa^{-1}\ln(1-\mu^2 q^2) > 0 \tag{6.6}$$

According to relations (6.6) and (6.3), the average velocity of the body M is equal to

$$V = \Delta x/T = -(\kappa T)^{-1}\ln(1-\mu^2\kappa^2 L^2) > 0 \tag{6.7}$$

It can be seen from formula (6.7) that the maximum average velocity V (with the assumed constraints) is reached for minimum T, with permissible constraints (3.4). Substituting the quantity u_2 from (6.4) into formula (3.3) for T, we obtain

$$T = 2L(1-\mu q)u_1^{-1}$$

The minimum value of T is reached for maximum $u_1 = U$, with the permissible constraints (3.4). Substituting the quantity u_2 from (6.4) into formula (3.3) for T, we obtain

$$T = 2L(1-\mu q)u_1^{-1}$$

The minimum value of T is reached for maximum $u_1 = U$, allowed by constraints (3.4). Then, according to expression (6.4), the constraint $u_2 < U$ is also satisfied. Substituting $u_1 = U$ into the relation for T and using equality (6.3), we obtain

$$T = 2L(1-\mu\kappa L)^{-1}U^{-1} \tag{6.8}$$

Substituting this value of T into Eq. (6.7), we express the maximum average velocity V (with the above constraints) for $\mu\kappa L < 1$ in terms of the initial parameters of the problem in the form

$$V = -U(2\kappa L)^{-1}(1-\mu\kappa L)\ln(1-\mu^2\kappa^2 L^2) > 0 \tag{6.9}$$

Hence, unlike the piecewise-linear resistance law, the average velocity is positive even in the case of an isotropic quadratic resistance. We recall that the velocity (6.9) is not the maximum value possible and corresponds to the case $v_0 = 0$.

7. Dry friction

As was noted in Section 4, the velocity $v(t)$ of the body M can change sign only at the instant τ_1, and inequalities (4.2) hold. In the case of dry friction, the motion of the body M in the interval $[0, T]$ may contain rest periods. Since the absolute value of the velocity $|v(t)|$, according to Eq. (1.3), does not increase in the intervals $(0, \tau_1)$ and (τ_1, T), the rest periods, during which $v(t) = 0$, can only be situated at the ends of these intervals.

Consequently, four versions of the motion of the body M in the period $[0, T]$ are possible in the case of dry friction (Fig. 3):

A – there are no rest intervals;
B – there is one non-zero rest interval (t_1, τ_1);
C – there is a single non-zero rest interval (t_2, T);
D – there are two non-zero rest intervals (t_1, τ_1) and (t_2, T).

Here t_1 and t_2 are the initial instants of the rest intervals, for which the following conditions are satisfied

$$0 < t_1 < \tau_1, \quad \tau_1 < t_2 < T$$

We will introduce the following notation

$$a_+ = f_+ g, \quad a_- = f_- g, \quad c = f_+/f_- = a_+/a_- \tag{7.1}$$

We will first consider Version A. Using Eqs. (1.3) and (1.6), conditions (4.1) and notation (7.1), we successively calculate the values of the velocity $v(t)$ at the instants when it experiences jumps. We have

$$\begin{aligned}
&\upsilon(0) = \upsilon_0, \quad \upsilon(+0) = \upsilon_0 - \mu u_1 \\
&\upsilon(\tau_1 - 0) = \upsilon_1 = \upsilon_0 - \mu u_1 + a_-\tau_1, \\
&\upsilon(\tau_1 + 0) = \upsilon_1 + \mu(u_1 + u_2) = \upsilon_0 + \mu u_2 + a_-\tau_1 \\
&\upsilon(T - 0) = \upsilon_2 = \upsilon_0 + \mu u_2 + a_-\tau_1 - a_+\tau_2 \\
&\upsilon(T) = \upsilon_2 - \mu u_2 = \upsilon_0 + a_-\tau_1 - a_+\tau_2
\end{aligned} \tag{7.2}$$

It follows from the last equality of (7.2) and the periodicity condition (1.10) that $a_-\tau_1 = a_+\tau_2$.
Hence, bearing Eqs. (3.3) and (7.1) in mind, we obtain

$$cu_1 = u_2 \tag{7.3}$$

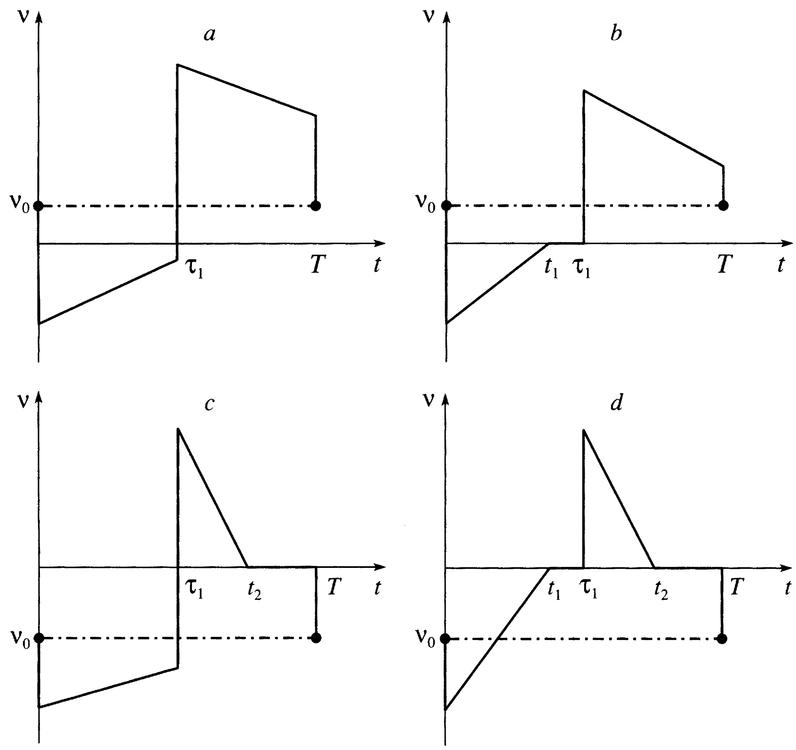

Fig. 3.

For Version A the following equalities must be satisfied

$$\upsilon(\tau_1 - 0) \le 0, \quad \upsilon(T-0) \ge 0$$

From these inequalities and relations (7.1)–(7.3) we obtain the conditions

$$-\mu c u_1 \le \upsilon_0 \le \mu u_1 - a\_\tau_1 \tag{7.4}$$

We will introduce the dimensionless variables

$$u_i = u_0 x_i, \quad u_0 = (La\_/\mu)^{1/2}, \quad i = 1, 2; \quad \upsilon_0 = \mu u_0 x_0, \qquad V = \mu u_0 F \tag{7.5}$$

and rewrite inequalities (7.4), expressing τ_1 using Eq. (3.3). We obtain

$$-cx_1 \le x_0 \le x_1 - x_1^{-1} \tag{7.6}$$

Since the left-hand side of inequality (7.6) does not exceed the right-hand side, the following conditions must be satisfied

$$x_1 \ge (1+c)^{-1/2}, \quad cx_1 = x_2 \tag{7.7}$$

The last equation of (7.7) follows from relations (7.3) and (7.5).

Hence, the dimensionless parameters x_0, x_1 and x_2 for Version A must satisfy conditions (7.6) and (7.7). The velocity $v(t)$ for Version A is a piecewise-linear function of time, which is uniquely defined by the quantities (7.2) at the instants of the jumps (see Fig. 3). Integrating this function over the period $[0, T]$, we determine the path $\Delta x = x(T) - x(0)$ traversed by the body M and the average velocity $V = \Delta x/T$. Carrying out the calculations and changing to the dimensioness variables (7.5), we obtain

$$F = x_0 + (2x_1)^{-1} \tag{7.8}$$

For Version B we obtain relations similar to (7.2)

$$\begin{aligned} &\upsilon(0) = \upsilon_0, \quad \upsilon(+0) = \upsilon_0 - \mu u_1; \quad \upsilon(t_1) = \upsilon_0 - \mu u_1 + a\_t_1 = 0 \\ &\upsilon(\tau_1 - 0) = 0, \quad \upsilon(\tau_1 + 0) = \mu(u_1 + u_2) \\ &\upsilon(T-0) = \mu(u_1 + u_2) - a_+\tau_2, \quad \upsilon(T) = \mu u_1 - a_+\tau_2 \end{aligned} \tag{7.9}$$

we obtain from the periodicity condition (1.10) and (7.9)

$$\upsilon_0 = \mu u_1 - a_+\tau_2 \tag{7.10}$$

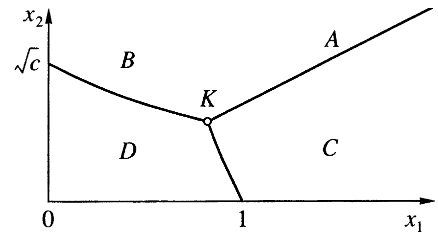

Fig. 4.

Substituting this expression into the third equation of (7.9) and taking notation (7.1) into account, we obtain

$$t_1 = c\tau_2 \tag{7.11}$$

The following conditions must be satisfied for Version B

$$t_1 < \tau_1, \quad v(T-0) \geq 0$$

These conditions, together with Eq. (7.10) and taking notation (7.1) and (7.5) into account, reduce to the form

$$x_0 = x_1 - cx_2^{-1}, \quad cx_1 < x_2, \quad (x_1 + x_2)x_2 \geq c \tag{7.12}$$

Versions C and D can be considered similarly. As a result, for each Version A–D, we obtain a set of conditions in the form of equalities and inequalities, imposed on the dimensionless parameters x_0, x_1 and x_2. Thus, for Version A we have conditions (7.6) and (7.7), while for Version B we have conditions (7.12). Moreover, we calculate the total displacement Δx of the system over the period of motion, for which we evaluate the integral over the period $[0, T]$ of the piecewise-linear function $v(t)$, shown in Fig. 3. We then determine the dimensionless average velocity V for each version, introduced by the last equality (7.5); for Version A it is given by formula (7.8). The overall results (the conditions imposed on the parameters x_0, x_1 and x_2 and the expressions for the function F) for all the versions A–D are presented below

$$A: cx_1 \leq x_0 \leq x_1 - \frac{1}{x_1}, \quad x_1 \geq (1+c)^{-1/2}, \quad cx_1 = x_2;$$

$$F = x_0 + \frac{1}{2x_1}$$

$$B: x_0 = x_1 - \frac{c}{x_2}, \quad cx_1 < x_2, \quad (x_1 + x_2)x_2 \geq c;$$

$$F = x_1 - \frac{c(1+c)x_1}{2(x_1 + x_2)x_2}$$

$$C: x_0 = -x_2, \quad cx_1 > x_2, \quad (x_1 + x_2)x_1 \geq 1;$$

$$F = \frac{(1+c)x_2}{2c(x_1 + x_2)x_1} - x_2$$

$$D: x_0 = -x_2, \quad (x_1 + x_2)x_1 < 1, \quad (x_1 + x_2)x_2 < c;$$

$$F = \frac{(1-c)(x_1 + x_2)x_1 x_2}{2c} \tag{7.13}$$

In the plane of the dimensionless parameters $x_1 > 0$ and $x_2 > 0$, versions A–D occur in the corresponding regions shown in Fig. 4 for $c < 1$. According to formulae (7.13) Version A occurs on the half-line

$$x_2 = cx_1, \quad x_1 \geq (1+c)^{-1/2}$$

which is the boundary between the regions B and C. The arcs of the corresponding hyperbolae

$$(x_1 + x_2)x_2 = c, \quad x_1 \leq (1+c)^{-1/2} \quad \text{and} \quad (x_1 + x_2)x_1 = c, \quad x_1 \geq (1+c)^{-1/2}$$

serve as the boundaries between the region D and the regions B and C. These arcs and the half-line $x_2 = cx_1$ meet at the point K with coordinates

$$x_1 = (1+c)^{-1/2}, \quad x_2 = c(1+c)^{-1/2}$$

Hence, for each pair of dimensionless parameters x_1 and x_2 or the dimensional parameters u_1 and u_2 we can determine which version of the periodic motion occurs. For Versions B–D from the values of x_1 and x_2 we can also determine, from formulae (7.13), the value of the parameter x_0 and the value of the dimensionless average velocity F. As regards Version A, it is first required to choose the quantity x_0 from the range indicated by the first relation of (7.13), and then calculate F. The values of the dimensional velocities u_1, u_2 and V are given by relations (7.5).

We will now determine the optimum values of the parameters u_1, u_2 and v_0, corresponding to the maximum average velocity V of periodic motion of the system. In dimensionless variables (7.5), this problem reduces to determining the dimensionless parameters x_1, x_2 and x_0, which yield the maximum value of the function F, defined by relations (7.13) in the different regions A–D with the constraints

$$0 < x_1 \le X, \quad 0 < x_2 \le X, \quad X = U/u_0 \tag{7.14}$$

These conditions follow from the limitations (3.4) imposed on the velocity of relative motion of the mass m.

A similar problem was considered previously in Refs. 5 and 6, where the additional condition $v_0 = 0$ was imposed. It can be seen from Eq. (7.13) that this condition, equivalent to the equality $x_0 = 0$, can only be satisfied for Versions A and B. This condition is not imposed in this paper, and hence the value of the average velocity turns out to be higher than in Refs. 5 and 6.

We will analyse the behaviour of the function F for Version A–D.

For Version A this function increases monotonically as x_0 increases. Consequently, the optimum value of the parameter x_0 for this version is given by the upper limit in inequality (7.6). According to relations (7.6) and (7.8) we have, for Version A,

$$x_0 = x_1 - x_1^{-1}, \quad F = x_1 - (2x_1)^{-1} \tag{7.15}$$

Since this function F increases monotonically as x_1 increases, it reaches its maximum for the greatest value of x_1, which, together with the equality $x_2 = cx_1$ (see (7.7)), satisfies inequalities (7.14).

For Version B, the function F, defined by formulae (7.13), increases as x_2 increases. In addition, we have $\partial^2 F/\partial x_1^2 > 0$ for all $x_1 > 0$ and $x_2 > 0$ in region B, so that F is a convex function of x_1. Hence, whereas the maximum is reached in the region B, this also occurs for the greatest value of x_2 on one of the boundaries of the permissible interval for the parameter x_1 with conditions (7.14).

In region C the function F decreases monotonically as x_1 increases. Consequently, it never reaches its maximum inside the region C; it can only occur on the boundary of regions C and D or on the straight line A (see Fig. 4).

In region D, the function F increases as x_1 and x_2 increase, if $c < 1$, and decreases as x_1 and x_2 increase, if $c > 1$. In the latter case we have $F < 0$.

Taking the above properties into account, we will determine the required maximum of the function F with constraints (7.14).

Suppose initially that $c < 1$ and the point H with coordinates $x_1 = x_2 = X$ lies inside the region D. It is easy to verify, using relations (7.13), that this occurs when $X\sqrt{c/2}$. In this case the required maximum of the function F is reached at the point H.

If, for $c < 1$, the point H lies outside region D, it lies in region B. The required maximum may be reached either at the point H, or at the point of intersection of the straight line $x_2 = X$ with the boundary between regions B and D. A comparison of the corresponding values of the function F, calculated from formulae (7.13), shows that, when $c < 1$, a maximum is always reached at the point $x_1 = x_2 = X$.

Suppose now that $c \ge 1$ and the condition $X \le c/\sqrt{2}$ is satisfied. In this case, in view of relations (7.13) and (7.15), we have $\dot{F} \le 0$ in all the regions A–D. When $c = 1$, the zero maximum of the function F occurs at any point of region D. When $c > 1$ we have in the limit $F \to -0$ when $x_1 \to 0$ or $x_2 \to 0$ in region D.

If $c \ge 1$ and $X > c/\sqrt{2}$, the maximum of the function F is reached on the line A when $x_1 = X/c$ and $x_2 = X$.

The results obtained can be summarized in the form of the following relations:

$$\begin{aligned}
&1)\ c < 1, \quad X < \sqrt{\frac{c}{2}}: (x_1, x_2) \in D, \quad x_1 = x_2 = X, \\
&x_0 = -X, \quad F = \frac{(1-c)X^3}{c} \\
&2)\ c < 1, \quad X \ge \sqrt{\frac{c}{2}}: (x_1, x_2) \in B, \quad x_1 = x_2 = X, \\
&x_0 = -X, \quad F = X - \frac{c(1+c)}{4X} \\
&3)\ c = 1, \quad X \le \frac{1}{\sqrt{2}}: (x_1, x_2) \in D, \quad F = 0 \\
&4)\ c > 1, \quad X \le \frac{c}{\sqrt{2}}: (x_1, x_2) \in D, \quad x_1 \to 0 \quad \text{or} \quad x_2 \to 0, \\
&x_0 \to 0, \quad F \to -0 \\
&5)\ c \ge 1, \quad X > \frac{c}{\sqrt{2}}: (x_1, x_2) \in A, \quad x_1 = \frac{X}{c}, \\
&x_2 = X, \quad x_0 = \frac{X^2 - c^2}{cX}, \quad F = \frac{2X^2 - c^2}{2cX}
\end{aligned} \tag{7.16}$$

The subdivision of the first quadrant of the plane of dimensionless parameters c and X, defined by relations (7.16), is shown in Fig. 5.

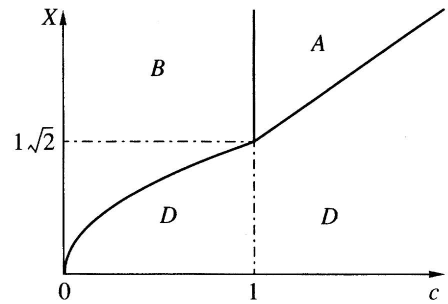

Fig. 5.

Hence, a positive average velocity $V > 0$ of a two-mass system can be achieved when $c < 1$ or when the two inequalities $c \geq 1$ and $X > c/\sqrt{2}$ are simultaneously satisfied.

When $c < 1$, the maximum of the velocity V is obtained in mode D with two rest intervals (t_1, τ_1) and (t_2, T) of the body M, if $X < c/\sqrt{2}$, and in mode B with a single rest interval (t_1, τ_1), if $X \geq c/\sqrt{2}$. In both cases the mass m moves with the maximum permissible relative velocity: $u_1 = u_2 = U$.

If the conditions $c > 1$ and $X > c/\sqrt{2}$ are satisfied, the maximum velocity V is reached in mode A, for which there are no rest intervals of the body M. Here we have $u_1 = U/c$ and $u_2 = U$.

Finally, for isotropic dry friction ($c = 1$), the maximum positive average velocity $V > 0$ is reached when $X > 1/\sqrt{2}$ in mode A, where here $u_1 = u_2 = U$.

8. Conclusion

For three forms of resistance forces we have calculated the periodic translational motion of a two-mass system consisting of the main body (the container), interacting with the medium, and an internal moving mass.

We have considered simple motions of the internal mass with respect to the container, called two-phase motion, and a corresponding piecewise-constant relative velocity of the internal mass. We have shown that as a result of simple motions of the internal mass with respect to the body, the system is displaced as a whole. The average velocity of the displacement over a period has been determined.

For piecewise-linear resistance and dry friction forces we have obtained the maximum possible velocity of motion of the system as a whole. In the case of a resistance proportional to the velocity of motion of the container, displacement of the system is only possible in the case of anisotropy, i.e., for different coefficients of resistance for motions forwards and backwards. In the case of a quadratic resistance and dry friction, displacement is also possible in the isotropic case.

The principle of motion considered in this paper was realized in a number of experimental models.[7–9] The internal displacements were obtained by means of a pendulum system[7] and a rotating mass. Mobile mini-robots, moving inside a tube, have been designed and successfully tested.[9] Experiments have confirmed the practical realizability of the principle described above for displacing bodies in resistant media.

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